TECHNISCHE UNIVERSITÄT DRESDEN

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Application of the generalized polynomial chaos expansion in stochastic simulations

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2^{nd} Probabilistic workshop, 8th–9th October 2009

Outline

Uncertainty quantification methods:

- Anti–optimization method
- Possibilitic methods:
  - Interval analysis
  - Sensitivity derivatives
  - Fuzzy set theory
  - Evidence theory and convex modeling
- Probabilistic methods:
  - Monte Carlo
  - Moment methods (FORM, SORM)
  - Response surface
  - Functional method (Polynomial chaos)

Fundamental idea: random space projection

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- Suppose we had a basis $\{\Psi_i(\xi)\}_{i=1}^m$ (e.g. ortogonal functions). Is it possible to identify $\{\alpha_i\}_{i=1}^n$ such that: $\alpha = \sum_{i=0}^m \alpha_i \Psi_i(\xi)$? Then

$$\alpha_i = \frac{1}{\Psi_i^2} \int_\Omega \alpha \Psi_j(\xi) d\mu(\xi)$$

  can be used as a ***mapping*** for projection the random space Ω to a deterministic space !

Fundamental theorems

Theorem 1

If $\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_n\}^T$ be a finite dimensional Gaussian random vector, then

$$H_2 = \left\{ \mathcal{X} | \mathcal{X} = x_0 + x_1\xi_1 + x_2\xi_2 + \ldots + x_N\xi_n,\ (x_0, x_1, \ldots, x_N)^T \in \mathbb{R}^n \right\},$$

is a Gaussian Hilbert space.

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Theorem 2

In general, if $\{\Psi_0(\boldsymbol{\xi}), \Psi_1(\boldsymbol{\xi}), \ldots, \Psi_M(\boldsymbol{\xi})\}$ be any random orthogonal basis in Hilbert space, then

$$H_2 = \left\{ \mathcal{X} | \mathcal{X} = x_0\Psi_0(\boldsymbol{\xi}) + x_1\Psi_1(\boldsymbol{\xi}) + x_2\Psi_2(\boldsymbol{\xi}) + \ldots + x_M\Psi_M(\boldsymbol{\xi}),\ (x_0, x_1, \ldots, x_M)^T \in \mathbb{R}^n \right\},$$

is also a Hilbert space.

Polynomial Chaos expansion (PCe)

Ω : sample space
$\mathcal{F}$: σ–algebra
P : probability measure

For probability space $(\Omega, \mathcal{F}, P)$, we define the Hilbert space $H_2(\Omega, \mathcal{F}, P)$ as

$$H_2 = \left\{ \mathcal{X} : \int_\Omega \left| \mathcal{X}(\theta)^2 \right| dP(\theta) < \infty \right\} , \quad \theta \in \Omega \tag{1}$$

An uncertain parameter $\mathcal{X} : \Omega \longrightarrow \mathbb{R}$ and $\mathcal{X} \in H_2$ can be represented by the *the generalized polynomial chaos expansion* of:

$$\begin{aligned} \mathcal{X} &= x_0 \Psi_0 + \sum_{i_1=1}^{\infty} x_{i_1} \Psi_1(\xi_{i_1}) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} x_{i_1 i_2} \Psi_2(\xi_{i_1}, \xi_{i_2}) \\ &+ \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} x_{i_1 i_2 i_3} \Psi_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) + \dots \end{aligned} \tag{2}$$

or shortly:

$$\mathcal{X} = \sum_{i=0}^{\infty} x_i \Psi_i(\boldsymbol{\xi}) , \qquad \text{and} \qquad \sum_{i=0}^{\infty} x_i^2 < \infty . \tag{3}$$

- $x_{i_1 i_2 \dots i_n}$ oder x_i : unknown deterministic PCe coefficients
- $\boldsymbol{\xi}$: vector of random variables
- Ψ's : orthogonal polynomial basis of random variables

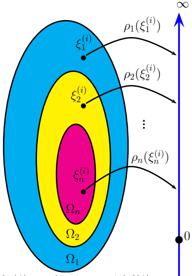
PCe in practice

$$\mathcal{X} \approx \sum_{i=0}^{N} x_i \Psi_i(\boldsymbol{\xi}) \overset{*}{\Longrightarrow} x_i = \frac{1}{h_i^2} \int_\Omega \langle \mathcal{X}\ ,\ \Psi_j(\boldsymbol{\xi}) \rangle \, d\mu(\boldsymbol{\xi}), \qquad j = 0, 1, \dots, N$$

$$\boldsymbol{\xi} = \{\xi_1, \xi_2, \dots \xi_n\}^T, \qquad \Omega = \Omega_1 \otimes \Omega_2 \otimes \dots \otimes \Omega_n, \quad \xi_i \in \Omega_i$$

$$\boldsymbol{w}(\mathbf{x}, t; \boldsymbol{\xi}) \approx \sum_{i=0}^{N} w_i(\mathbf{x}, t) \Psi_i(\boldsymbol{\xi}) \overset{*}{\Longrightarrow} w_i(\mathbf{x}, t) = \frac{1}{h_i^2} \int_\Omega \langle \boldsymbol{w}(\mathbf{x}, t; \boldsymbol{\xi})\ ,\ \Psi_j(\boldsymbol{\xi}) \rangle \, d\mu(\boldsymbol{\xi})$$

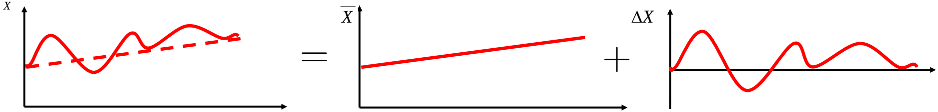
- $\boldsymbol{w}(\mathbf{x}, t; \boldsymbol{\xi})$: stochastic process
- $w_i(\mathbf{x}, t)$: unknown deterministic functions
- $E[\Psi_i, \Psi_j] = h_i^2 \delta_{ij}$: expected value
- $d\mu(\boldsymbol{\xi})$: probability measure
- $\rho(\boldsymbol{\xi})$: density function
- h_i^2 : polynomial norm
- δ_{ij} : Kronecker delta



$$d\mu(\boldsymbol{\xi}) = \boldsymbol{\rho}(\xi_1, \xi_2, \dots, \xi_n) d(\boldsymbol{\xi})$$
$$\boldsymbol{\rho}(\xi_1, \xi_2, \dots, \xi_n) = \rho_1(\xi_1)\rho_2(\xi_1)\dots\rho_n(\xi_1)$$

* stochastic Galerkin projection

Graphical representation of the PCe ($\mathcal{X} = \bar{\mathcal{X}} + \Delta\mathcal{X}$)



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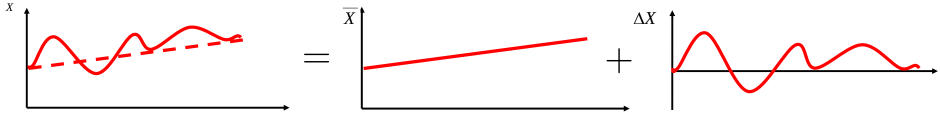
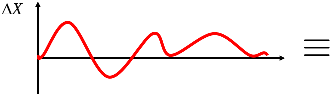
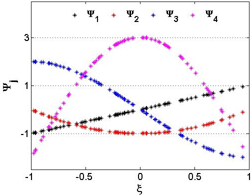


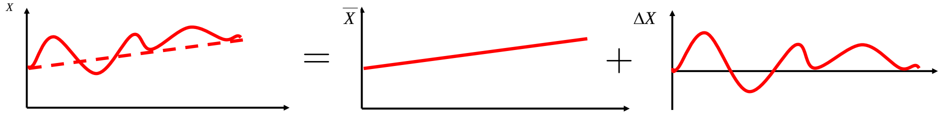
if using standard random polynomials is possible:





$$\Delta\mathcal{X} = \sum_{j=1}^{N} x_j \Psi_j(\xi)$$

Graphical representation of the PCe ($\mathcal{X} = \bar{\mathcal{X}} + \Delta\mathcal{X}$)



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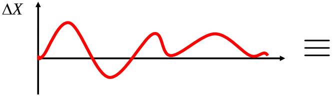
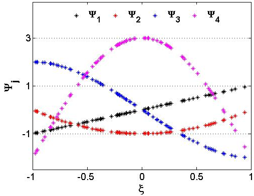
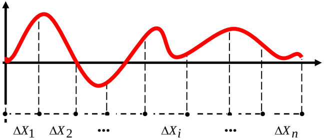




$$\Delta\mathcal{X} = \sum_{j=1}^{N} x_j \Psi_j(\xi)$$

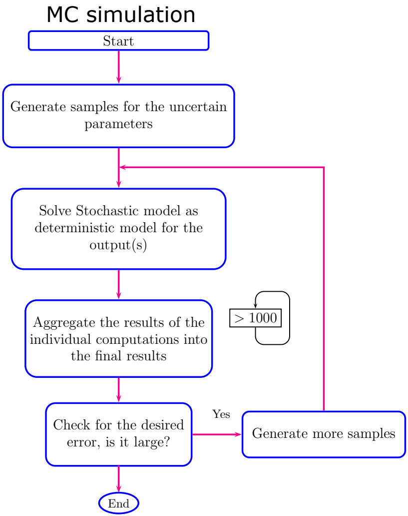
if not, use multi–element PCe:

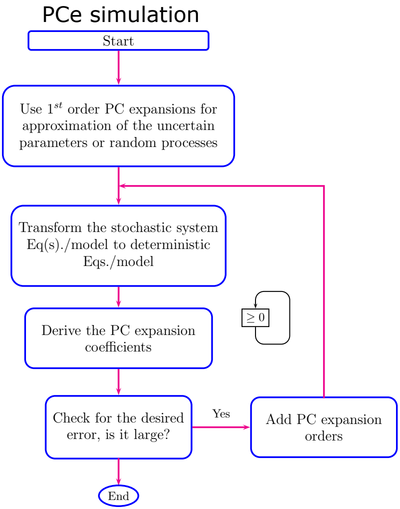


$$\Delta\mathcal{X}_i = \sum_{j=1}^{N} \bar{x}_{i_j} \bar{\Psi}_j(\xi_i)$$

$$\Delta\mathcal{X} = \sum_{i=1}^{n} \Delta\mathcal{X}_i$$

MC and PCe


MC simulation
Start
Generate samples for the uncertain parameters
Solve Stochastic model as deterministic model for the output(s)
Aggregate the results of the individual computations into the final results
> 1000
Check for the desired error, is it large?
Yes
Generate more samples
End


PCe simulation
Start
Use 1st order PC expansions for approximation of the uncertain parameters or random processes
Transform the stochastic system Eq(s)./model to deterministic Eqs./model
Derive the PC expansion coefficients
≥ 0
Check for the desired error, is it large?
Yes
Add PC expansion orders
End

Classic example

$$C = A + B\,, \quad A \in \Omega_1, \quad B \in \Omega_2$$

$$A = \sum_{i=0}^{N_1} a_i \Psi_{1_i}(\xi_1)\,, \quad B = \sum_{j=0}^{N_2} b_j \Psi_{2_j}(\xi_2)\,,$$

$$C = \sum_{k=0}^{N_3} c_k \Psi_k(\xi_1, \xi_2)\,, \quad \Psi(\xi_1, \xi_2) = \Psi_1(\xi_1) \otimes \Psi_2(\xi_2)\,.$$

$$c_k = \frac{1}{e_{kk}^2} \left[\sum_{i=0}^{N_1} a_i e_{ip} + \sum_{j=0}^{N_2} b_j e_{jp} \right]\,, \quad p = 0, \ldots, N_3$$

$$e_{ip} = \int_{\Omega_1} \int_{\Omega_2} \Psi_{1_i}(\xi_1) \Psi_p(\xi_1, \xi_2) d\mu(\xi_1, \xi_2)\,,$$

$$e_{jp} = \int_{\Omega_1} \int_{\Omega_2} \Psi_{2_j}(\xi_2) \Psi_p(\xi_1, \xi_2) d\mu(\xi_1, \xi_2)\,,$$

$$e_{kk}^2 = \int_{\Omega_1} \int_{\Omega_2} \Psi_k^2(\xi_1, \xi_2) d\mu(\xi_1, \xi_2)\,,$$

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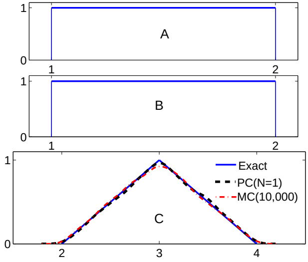


First order Legendre–PCe: ($N_1 = N_2 = 1$)

PCe and FE method

FE Model (stochastic): $KU = F$

PCe for uncertain parameters: $K,\ F$

$$K(\xi_1) = \sum_{i=0}^{N_k} [k]_i \Psi_{1_i}(\xi_1) = [k]_0 \Psi_{1_0}(\xi_1) + [k]_1 \Psi_{1_1}(\xi_1) + \ldots + [k]_{N_k} \Psi_{1_{N_k}}(\xi_1)$$

$$F(\xi_2) = \sum_{i=0}^{N_f} \{f\}_i \Psi_{2_i}(\xi_2) = \{f\}_0 \Psi_{2_0}(\xi_2) + \{f\}_1 \Psi_{2_1}(\xi_2) + \ldots + \{f\}_{N_f} \Psi_{2_{N_f}}(\xi_2)$$

System response: $U(\xi_1, \xi_2)$

$$\begin{aligned} U(\xi_1,\xi_2) &= \sum_{j=0}^{N_u} \{u\}_j \Psi_j(\xi_1,\xi_2) = \{u\}_0 \Psi_0(\xi_1,\xi_2) + \{u\}_1 \Psi_1(\xi_1,\xi_2) + \ldots \\ &+ \{u\}_{N_u} \Psi_{N_u}(\xi_1,\xi_2), \qquad \boldsymbol{\Psi} = \boldsymbol{\Psi}_1 \otimes \boldsymbol{\Psi}_2 \end{aligned}$$

PCe and FE method

PCe representation of FE Model: $(\boldsymbol{K}\boldsymbol{U} = \boldsymbol{F})$

$$\sum_{i=0}^{N_k}[\boldsymbol{k}]_i\Psi_{1_i}(\xi_1)\sum_{j=0}^{N_u}\{\boldsymbol{u}\}_j\Psi_j(\xi_1,\xi_2) = \sum_{i=0}^{N_f}\{\boldsymbol{f}\}_i\Psi_{2_i}(\xi_2)$$

Stochastic Galerkin projection with test functions $\Psi_m(\xi_1,\xi_2)$ leads to:

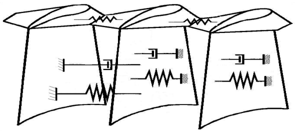
$$\boxed{\hat{\boldsymbol{K}}\boldsymbol{C}\hat{\boldsymbol{U}}=\hat{\boldsymbol{F}}}$$

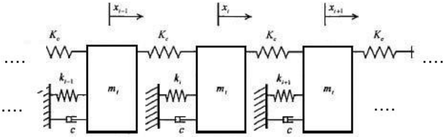
$$\hat{\boldsymbol{U}} = \{\{\boldsymbol{u}\}_0, \{\boldsymbol{u}\}_1, \ldots, \{\boldsymbol{u}\}_{N_u}\}^T$$

$$\hat{\boldsymbol{K}} = \begin{bmatrix} [\boldsymbol{k}]_0 & [\boldsymbol{k}]_1 & \ldots & [\boldsymbol{k}]_{N_k} \\ [\boldsymbol{k}]_0 & [\boldsymbol{k}]_1 & \ldots & [\boldsymbol{k}]_{N_k} \\ \ldots & \ldots & \ldots & \ldots \\ [\boldsymbol{k}]_0 & [\boldsymbol{k}]_1 & \ldots & [\boldsymbol{k}]_{N_k} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} C_{000} & C_{010} & \ldots & C_{0\boldsymbol{N_u}0} \\ C_{101} & C_{111} & \ldots & C_{1\boldsymbol{N_u}1} \\ \ldots & \ldots & \ldots & \ldots \\ C_{N_k0m} & C_{N_k1m} & \ldots & C_{N_k\boldsymbol{N_u}m} \end{bmatrix},$$

$$C_{ijm} = \int_{\Omega_1}\int_{\Omega_2}\ldots\int_{\Omega_n}\Psi_{1_i}\Psi_j\Psi_m d\mu(\xi), \qquad \hat{\boldsymbol{F}} = \langle \boldsymbol{F}, \Psi_m\rangle$$

PCe model of a bladed disk assembly





$$m_t\ddot{x}_i + c\dot{x}_i + k_i x_i + K_c(x_i - x_{i+1}) + K_c(x_i - x_{i-1}) = f_i$$

$$x_i = x_i(t;\xi) \qquad i = 1, 2, \ldots, n$$

and order excitation:

$$f_i = f_0 e^{(\omega_i t - \psi_{i,r})}, \qquad \psi_{i,r} = \frac{2\pi r(i-1)}{n}, \qquad r = 0, 1, 2, \ldots, n$$

for stochastic steady state motion of the blades:

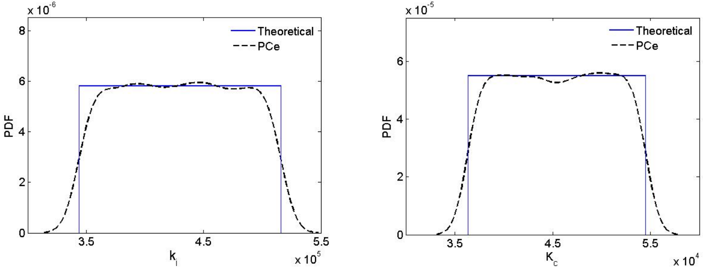
$$x_i(t;\xi) = A_i(\xi) e^{-i\omega t}$$

Numerical results – rotor of 24 blades

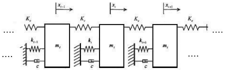
Assuming K_c and k_i as uncertain parameters with uniform PDF (with %20 uncertainty from the mean values).

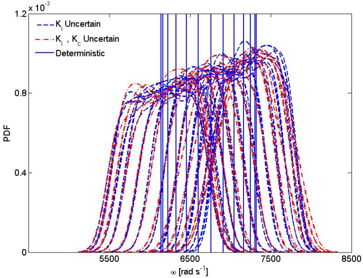
PCe representation of k_i and K_c:

$$k_i = \sum_{j=0}^{1} \tilde{k}_{i_j} \phi_{i_j}(\xi_1)\,, \qquad K_c = \sum_{j=0}^{1} \tilde{k}_{c_j} \phi_j(\xi_1)\,, \qquad \xi_1 \in U[-1,1]$$



Numerical results – natural frequencies





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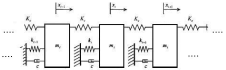

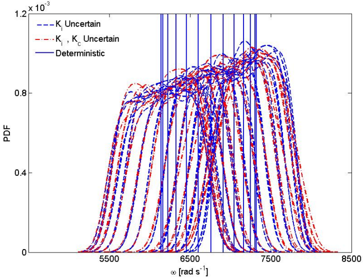

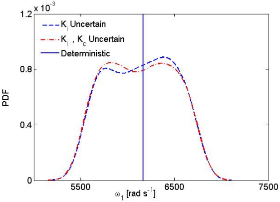

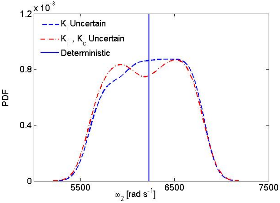

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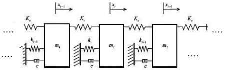
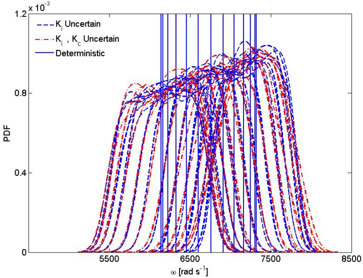
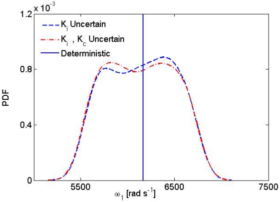
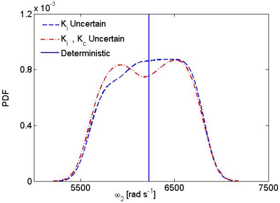
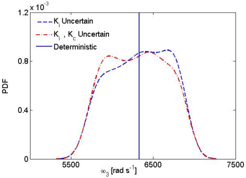
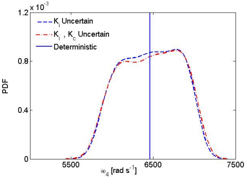


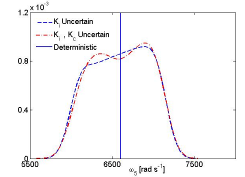




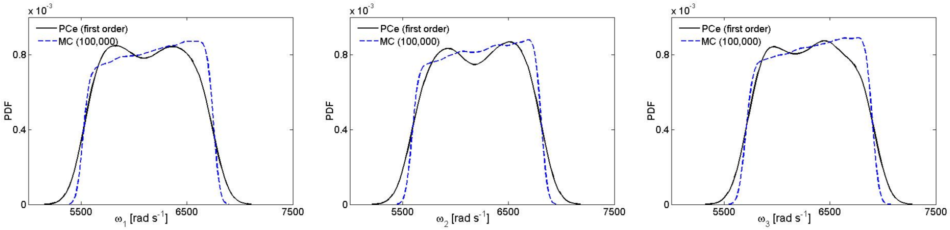








Comparison with MC simulations



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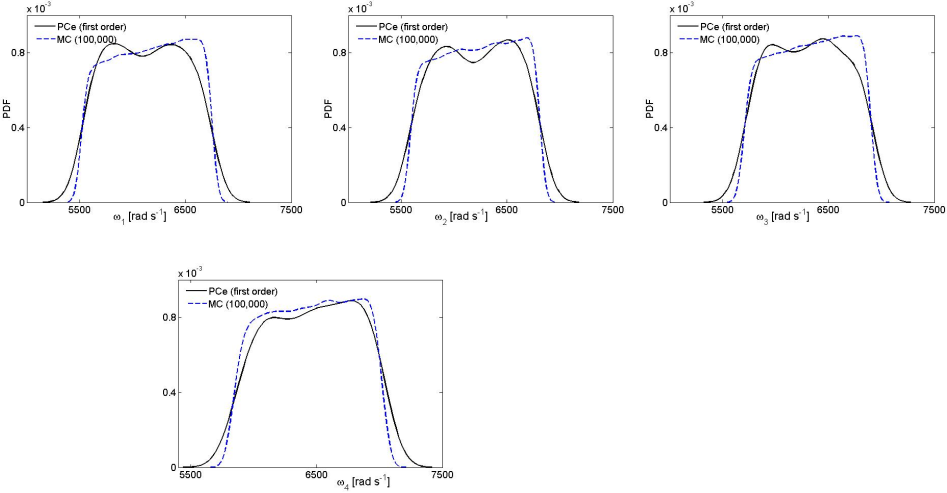

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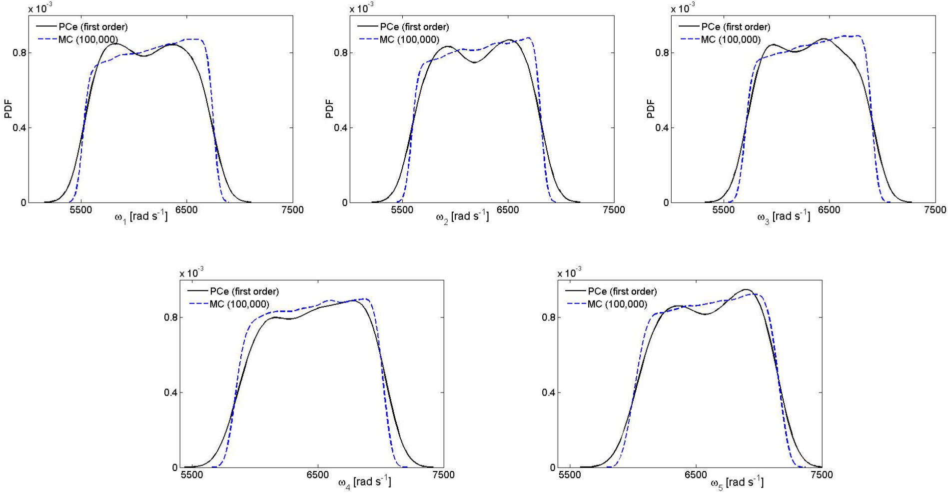
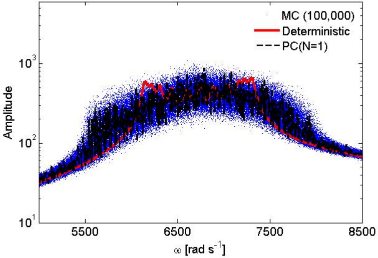


Frequency response functions – FRF



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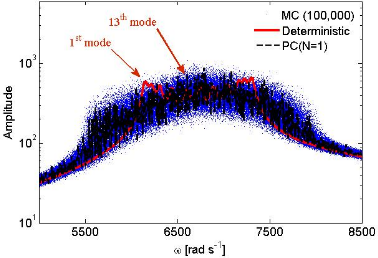
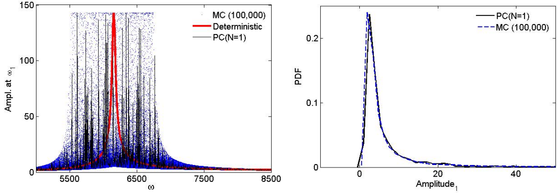
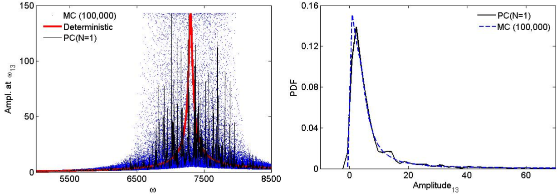


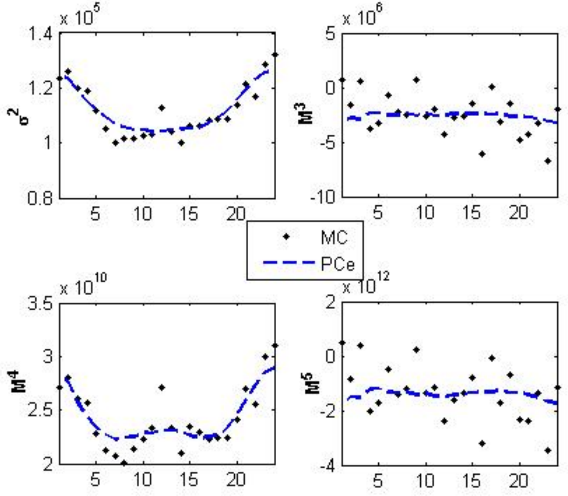


1^{st} mode



13^{th} mode

Statistical moments of natural frequencies



Horizontal axis: mode No

Conclusions

- The PCe method can be considered as a **mapping** between stochastic and deterministic spaces.
- The PCe is a reliable method for **closed from** representation of system uncertainties.
- It is shown that the PCe model can be easily combined with FE method for stochastic simulation of complex systems.
- The PCe model is reasonable accurate in compare with MC simulation and more **time efficient**.
- The convergency of **higher order statistical moments** can not be guaranteed by PCe method.
- The application of the method for general uncertainty representation in stochastic simulation is an actual research topic in **IFKM**.

Thank you for your attention!